BCS Theory of Superconductivity

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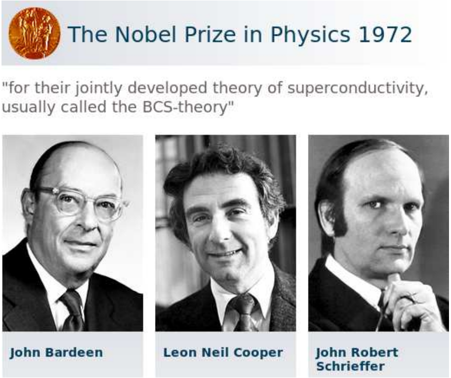
June 11, 2007

What is BCS Theory?



Original publication: Phys. Rev. **108**, 1175 (1957)

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- First “working” microscopic theory for superconductors.
- It’s a mean-field theory.
- In it’s original form only applied for conventional superconductors.

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Outline

1. Cooper-Pairs
   - Formation of Pairs
   - Origin of Attractive Interaction
2. BCS Theory
   - The model Hamiltonian
   - Bogoliubov-Valatin-Transformation
   - Calculation of the condensation energy
3. Finite Temperatures
   - Excitation Energies and the Energy Gap
   - Determination of T_c
   - Temperature dependence of the energy gap
   - Thermodynamic quantities

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Formation of Pairs

Let's assume the following things:

- Consider a material with a filled Fermi sea at $T = 0$.
- Add two more electrons that
  - interact attractively with each other but
  - don't interact with the other electrons except via Pauli-prinziple.

Formation of Pairs

Look for the groundstate wavefunction for the two added electrons, which has zero momentum:

$$\Psi_0(\mathbf{r}_1, \mathbf{r}_2) = \sum_{\mathbf{k}} \left(g_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}_1} e^{-i\mathbf{k}\cdot\mathbf{r}_2} \right) (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$$

The total wavefunction has to be antisymmetric with respect to exchange of the two electrons. The spin part is antisymmetric and therefore the spacial part has to be symmetric.

$$\Rightarrow g_{\mathbf{k}} \stackrel{!}{=} g_{-\mathbf{k}}.$$

Formation of Pairs

Inserting this into the Schrödinger equation of the problem leads to the following equation for the determination of the coefficients $g_{\mathbf{k}}$ and the energy eigenvalue E:

$$(E - 2\epsilon_{\mathbf{k}})g_{\mathbf{k}} = \sum_{k>k_F} V_{\mathbf{k}\mathbf{k}'} g_{\mathbf{k}'},$$

where

$$V_{\mathbf{k}\mathbf{k}'} = \frac{1}{\Omega} \int V(\mathbf{r}) e^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{r}} d\mathbf{r}$$

($\mathbf{r}$: distance between the two electrons, Ω: normalization volume, $\epsilon_{\mathbf{k}}$: unperturbated plane-wave energies).

Formation of Pairs

Since it is hard to analyze the situation for general $V_{\mathbf{k}\mathbf{k}'}$, assume:

$$V_{\mathbf{k}\mathbf{k}'} = \begin{cases} -V & , E_F < \epsilon_{\mathbf{k}} < E_F + \hbar\omega_c \\ 0 & , \text{otherwise} \end{cases}$$

with $\hbar\omega_c$ a cutoff energy away from E_F.

Formation of Pairs

With this approximation we get:

$$\begin{aligned}\frac{1}{V} &= \sum_{k>k_F} \frac{1}{2\epsilon_{\mathbf{k}} - E} = N(0) \int_{E_F}^{E_F+\hbar\omega_c} \frac{d\epsilon}{2\epsilon - E} \\ &= \frac{1}{2} N(0) \ln\left(\frac{2E_F - E + 2\hbar\omega_c}{2E_F - E}\right).\end{aligned}$$

If $N(0)V \ll 1$, we can solve approximativly for the energy E

$$E \approx 2E_F - 2\hbar\omega_c e^{-\frac{2}{N(0)V}} < 2E_F.$$

Origin of Attractive Interaction

Negative terms come in when one takes the motion of the ion cores into account, e.g. considering electron-phonon interactions. The physical idea is that

- the first electron polarizes the medium by attracting positive ions;
- these excess positive ions in turn attract the second electron, giving an effective attractive interaction between the electrons.

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BCS Theory

Having seen that the Fermi sea is unstable against the formation of a bound Cooper pair when the net interaction is attractive, we must then expect pairs to condense until an equilibrium point is reached.

We need a smart way to write down antisymmetric wavefunctions for many electrons. This will be done in the language of **second quantization.**

BCS Theory

Introduce the creation operator $c^\dagger_{\mathbf{k}\sigma}$, which creates an electron of momentum $\mathbf{k}$ and spin σ, and the correspondig annihilation operator $c_{\mathbf{k}\sigma}$. These operators obey the standard anticommutation relations for fermions:

$$\{c_{\mathbf{k}\sigma}, c^\dagger_{\mathbf{k}'\sigma'}\} \equiv c_{\mathbf{k}\sigma} c^\dagger_{\mathbf{k}'\sigma'} + c^\dagger_{\mathbf{k}'\sigma'} c_{\mathbf{k}\sigma} = \delta_{\mathbf{k}\mathbf{k}'}\delta_{\sigma\sigma'}$$

$$\{c_{\mathbf{k}\sigma}, c_{\mathbf{k}'\sigma'}\} = 0 = \{c^\dagger_{\mathbf{k}\sigma}, c^\dagger_{\mathbf{k}'\sigma'}\}.$$

Additionally the particle number operator $n_{\mathbf{k}\sigma}$ is defined by

$$n_{\mathbf{k}\sigma} \equiv c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma}.$$

The model Hamiltonian

We start with the so-called

pairing-hamiltonian

$$\mathcal{H} = \sum_{\mathbf{k}\sigma} \epsilon_{\mathbf{k}} n_{\mathbf{k}\sigma} + \sum_{\mathbf{kl}} V_{\mathbf{kl}} c^{\dagger}_{\mathbf{k}\uparrow} c^{\dagger}_{-\mathbf{k}\downarrow} c_{-\mathbf{l}\downarrow} c_{\mathbf{l}\uparrow},$$

presuming that it includes the terms that are decisive for superconductivity, although it omits many other terms which involve electrons not paired as $(\mathbf{k}\uparrow, -\mathbf{k}\downarrow)$.

The model Hamiltonian

We then add a term $-\mu\mathcal{N}$, where μ is the chemical potential, leading to

$$\mathcal{H} - \mu\mathcal{N} = \sum_{\mathbf{k}\sigma} \xi_{\mathbf{k}} n_{\mathbf{k}\sigma} + \sum_{\mathbf{kl}} V_{\mathbf{kl}} c^{\dagger}_{\mathbf{k}\uparrow} c^{\dagger}_{-\mathbf{k}\downarrow} c_{-\mathbf{l}\downarrow} c_{\mathbf{l}\uparrow}.$$

The inclusion of this factor is mathematically equivalent to taking the zero of kinetic energy to be at μ (or E_F).

Bogoliubov-Valatin-Transformation

Define:

$$b_{\mathbf{k}} \equiv \langle c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow} \rangle$$

Because of the large number of particles involved, the fluctuations of $c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow}$ about these expectations values $b_{\mathbf{k}}$ should be small. Therefor express such products of operators formally as

$$c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow} = b_{\mathbf{k}} + (c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow} - b_{\mathbf{k}})$$

and neglect quantities which are bilinear in the presumably small fluctuation term in parentheses.

Bogoliubov-Valatin-Transformation

Inserting this in our pairing Hamiltonian we obtain the so-called

model-hamiltonian

$$\mathcal{H}_M - \mu\mathcal{N} = \sum_{\mathbf{k}\sigma} \xi_{\mathbf{k}} c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} + \sum_{\mathbf{kl}} V_{\mathbf{kl}} (c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} b_{\mathbf{l}} + b^*_{\mathbf{k}} c_{-\mathbf{l}\downarrow} c_{\mathbf{l}\uparrow} - b^*_{\mathbf{k}} b_{\mathbf{l}})$$

where the $b_{\mathbf{k}}$ are to be determined self-consistently.

Bogoliubov-Valatin-Transformation

Defining further

$$\Delta_{\mathbf{k}} = -\sum_{\mathbf{l}} V_{\mathbf{kl}} b_{\mathbf{l}} = -\sum_{\mathbf{l}} V_{\mathbf{kl}} \langle c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow} \rangle$$

leads to the following form of the

model-hamiltonian

$$\mathcal{H}_M - \mu\mathcal{N} = \sum_{\mathbf{k}\sigma} \xi_{\mathbf{k}} c^{\dagger}_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} - \sum_{\mathbf{k}} (\Delta_{\mathbf{k}} c^{\dagger}_{\mathbf{k}\uparrow} c^{\dagger}_{-\mathbf{k}\downarrow} + \Delta^{*}_{\mathbf{k}} c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow} - \Delta_{\mathbf{k}} b^{*}_{\mathbf{k}})$$

Bogoliubov-Valatin-Transformation

This hamiltonian can be diagonalized by a suitable linear transformation to define new Fermi operators $\gamma_{\mathbf{k}}$:

Bogoliubov-Valatin-Transformation

$$\begin{aligned} c_{\mathbf{k}\uparrow} &= u_{\mathbf{k}}^{*}\gamma_{\mathbf{k}\uparrow} + v_{\mathbf{k}}\gamma_{-\mathbf{k}\downarrow}^{\dagger} \\ c_{-\mathbf{k}\downarrow}^{\dagger} &= -v_{\mathbf{k}}^{*}\gamma_{\mathbf{k}\uparrow} + u_{\mathbf{k}}\gamma_{-\mathbf{k}\downarrow}^{\dagger} \end{aligned}$$

with $|u_{\mathbf{k}}|^2 + |v_{\mathbf{k}}|^2 = 1$. Our "job" is now to determine the values of $v_{\mathbf{k}}$ and $u_{\mathbf{k}}$.

Bogoliubov-Valatin-Transformation

Inserting these operators in the model-hamiltonian gives

$$\begin{aligned}\mathcal{H}_M - \mu\mathcal{N} \;=\; & \sum_{\mathbf{k}} \xi_{\mathbf{k}} \Big((|u_{\mathbf{k}}|^2 - |v_{\mathbf{k}}|^2)(\gamma^\dagger_{\mathbf{k}\uparrow}\gamma_{\mathbf{k}\uparrow} + \gamma^\dagger_{-\mathbf{k}\downarrow}\gamma_{-\mathbf{k}\downarrow}) \\ & + 2|v_{\mathbf{k}}|^2 + 2u^*_{\mathbf{k}} v^*_{\mathbf{k}} \gamma_{-\mathbf{k}\downarrow}\gamma_{\mathbf{k}\uparrow} + 2u_{\mathbf{k}} v_{\mathbf{k}} \gamma^\dagger_{\mathbf{k}\uparrow}\gamma^\dagger_{-\mathbf{k}\downarrow} \Big) \\ & + \sum_{\mathbf{k}} \Big((\Delta_{\mathbf{k}} u_{\mathbf{k}} v^*_{\mathbf{k}} + \Delta^*_{\mathbf{k}} u^*_{\mathbf{k}} v_{\mathbf{k}})(\gamma^\dagger_{\mathbf{k}\uparrow}\gamma_{\mathbf{k}\uparrow} + \gamma^\dagger_{-\mathbf{k}\downarrow}\gamma_{-\mathbf{k}\downarrow} - 1) \\ & + (\Delta_{\mathbf{k}} v^{*2}_{\mathbf{k}} - \Delta^*_{\mathbf{k}} u^{*2}_{\mathbf{k}})\gamma_{-\mathbf{k}\downarrow}\gamma_{\mathbf{k}\uparrow} \\ & + (\Delta^*_{\mathbf{k}} v^2_{\mathbf{k}} - \Delta_{\mathbf{k}} u^2_{\mathbf{k}})\gamma^\dagger_{\mathbf{k}\uparrow}\gamma^\dagger_{-\mathbf{k}\downarrow} + \Delta_{\mathbf{k}} b^*_{\mathbf{k}} \Big) .\end{aligned}$$

Bogoliubov-Valatin-Transformation

Choose $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ so that the coefficients of $\gamma_{-\mathbf{k}\downarrow}\gamma_{\mathbf{k}\uparrow}$ and $\gamma^{\dagger}_{\mathbf{k}\uparrow}\gamma^{\dagger}_{-\mathbf{k}\downarrow}$ vanish.

$$\Rightarrow \quad 2\xi_{\mathbf{k}} u_{\mathbf{k}} v_{\mathbf{k}} + \Delta^*_{\mathbf{k}} v^2_{\mathbf{k}} - \Delta_{\mathbf{k}} u^2_{\mathbf{k}} = 0 \qquad \Big| \cdot \frac{\Delta^*_{\mathbf{k}}}{u^2_{\mathbf{k}}}$$

$$\Rightarrow \quad \left(\frac{\Delta^*_{\mathbf{k}} v_{\mathbf{k}}}{u_{\mathbf{k}}}\right)^2 + 2\xi_{\mathbf{k}}\left(\frac{\Delta^*_{\mathbf{k}} v_{\mathbf{k}}}{u_{\mathbf{k}}}\right) - |\Delta_{\mathbf{k}}|^2 = 0$$

$$\Rightarrow \quad \frac{\Delta^*_{\mathbf{k}} v_{\mathbf{k}}}{u_{\mathbf{k}}} = \sqrt{\xi^2_{\mathbf{k}} + |\Delta_{\mathbf{k}}|^2} - \xi_{\mathbf{k}} \equiv E_{\mathbf{k}} - \xi_{\mathbf{k}}$$

Bogoliubov-Valatin-Transformation

This gives us an equation for the $v_{\mathbf{k}}$ and $u_{\mathbf{k}}$ as

$$|v_{\mathbf{k}}|^2 = 1 - |u_{\mathbf{k}}|^2 = \frac{1}{2}\left(1 - \frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}}\right).$$

The BCS ground state

BCS took as their form for the ground state

$$|\Psi_G\rangle = \prod_{\mathbf{k}} (u_{\mathbf{k}} + v_{\mathbf{k}} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow}) |0\rangle$$

where $|u_{\mathbf{k}}|^2 + |v_{\mathbf{k}}|^2 = 1$. This form implies that the probability of the pair $(\mathbf{k}\uparrow, -\mathbf{k}\downarrow)$ being occupied is $|v_{\mathbf{k}}|^2$, whereas the probability that it is unoccupied is $|u_{\mathbf{k}}|^2 = 1 - |v_{\mathbf{k}}|^2$.

Note: $|\Psi_G\rangle$ is the vacuum state for the γ operators, e.g.

$$\gamma_{\mathbf{k}\uparrow} |\Psi_G\rangle = 0 = \gamma_{-\mathbf{k}\downarrow} |\Psi_G\rangle$$

Calculation of the condensation energy

We can now calculate the groundstate energy to be

$$\begin{aligned}\langle\Psi_G|\,\mathcal{H}-\mu\mathcal{N}\,|\Psi_G\rangle &= 2\sum_{\mathbf{k}}\xi_{\mathbf{k}}v_{\mathbf{k}}^2+\sum_{\mathbf{kl}}V_{\mathbf{kl}}u_{\mathbf{k}}v_{\mathbf{k}}u_{\mathbf{l}}v_{\mathbf{l}} \\ &= \sum_{\mathbf{k}}\left(\xi_{\mathbf{k}}-\frac{\xi_{\mathbf{k}}^2}{E_{\mathbf{k}}}\right)-\frac{\Delta^2}{V}\end{aligned}$$

The energy of the normal state at $T=0$ corresponds to the BCS state with $\Delta=0$ and $E_{\mathbf{k}}=|\xi_{\mathbf{k}}|$. Thus

$$\langle\Psi_n|\,\mathcal{H}-\mu\mathcal{N}\,|\Psi_n\rangle=\sum_{|\mathbf{k}|<k_F}2\xi_{\mathbf{k}}$$

Calculation of the condensation energy

Thus, the condensation energy is given by

$$\begin{aligned}
\langle E \rangle_s - \langle E \rangle_n &= \sum_{|\mathbf{k}|>k_F} \left(\xi_{\mathbf{k}} - \frac{\xi_{\mathbf{k}}^2}{E_{\mathbf{k}}} \right) + \sum_{|\mathbf{k}|<k_F} \left(-\xi_{\mathbf{k}} - \frac{\xi_{\mathbf{k}}^2}{E_{\mathbf{k}}} \right) - \frac{\Delta^2}{V} \\
&= 2 \sum_{|\mathbf{k}|>k_F} \left(\xi_{\mathbf{k}} - \frac{\xi_{\mathbf{k}}^2}{E_{\mathbf{k}}} \right) - \frac{\Delta^2}{V} \\
&= \left(\frac{\Delta^2}{V} - \frac{1}{2} N(0) \Delta^2 \right) - \frac{\Delta^2}{V} = -\frac{1}{2} N(0) \Delta^2
\end{aligned}$$

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Excitation Energies and the Energy Gap

With the above choice of the $u_\mathbf{k}$ and $v_\mathbf{k}$, the model-hamiltonian becomes

$$\mathcal{H}_M - \mu\mathcal{N} = \sum_\mathbf{k}(\xi_\mathbf{k} - E_\mathbf{k} + \Delta_\mathbf{k} b^*_\mathbf{k}) + \sum_\mathbf{k} E_\mathbf{k}(\gamma^\dagger_{\mathbf{k}\uparrow}\gamma_{\mathbf{k}\uparrow} + \gamma^\dagger_{-\mathbf{k}\downarrow}\gamma_{-\mathbf{k}\downarrow}).$$

$$E_\mathbf{k} = \sqrt{\Delta^2_\mathbf{k} + \xi^2_\mathbf{k}}$$

Excitation Energies and the Energy Gap

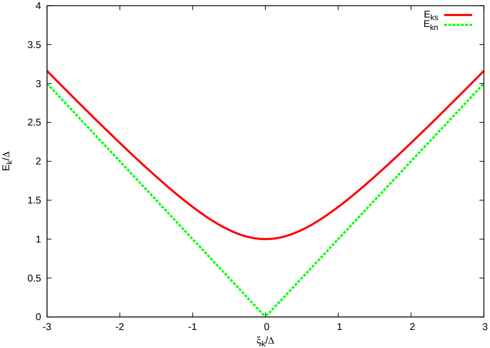


Figure: Energies of elementary excitations in the normal and superconducting states as functions of $\xi_{\mathbf{k}}$.

Excitation Energies and the Energy Gap

Inserting the γ operators in the definition of $\Delta_{\mathbf{k}}$ gives

$$\begin{aligned}\Delta_{\mathbf{k}} &= -\sum_{\mathbf{l}} V_{\mathbf{kl}} \langle c_{-\mathbf{l}\downarrow} c_{\mathbf{l}\uparrow} \rangle \\ &= -\sum_{\mathbf{l}} V_{\mathbf{kl}} u_{\mathbf{l}}^* v_{\mathbf{l}} \left\langle 1 - \gamma_{\mathbf{l}\uparrow}^\dagger \gamma_{\mathbf{l}\uparrow} - \gamma_{-\mathbf{l}\downarrow}^\dagger \gamma_{-\mathbf{l}\downarrow} \right\rangle \\ &= -\sum_{\mathbf{l}} V_{\mathbf{kl}} u_{\mathbf{l}}^* v_{\mathbf{l}} (1 - 2f(E_{\mathbf{l}})) \\ &= -\sum_{\mathbf{l}} V_{\mathbf{kl}} \frac{\Delta_{\mathbf{l}}}{2E_{\mathbf{l}}} \tanh \frac{\beta E_{\mathbf{l}}}{2}\end{aligned}$$

Excitation Energies and the Energy Gap

Using again the approximated potential $V_{\mathbf{kl}} = -V$, we have $\Delta_{\mathbf{k}} = \Delta_{\mathbf{l}} = \Delta$ and therefor

$$\frac{1}{V} = \frac{1}{2} \sum_{\mathbf{k}} \frac{\tanh(\beta E_{\mathbf{k}}/2)}{E_{\mathbf{k}}}.$$

This formula determines the critical temperature T_c!

Determination of T_c

The critical temperature T_c is the temperature at which $\Delta_{\mathbf{k}} \to 0$ and thus $E_{\mathbf{k}} \to \xi_{\mathbf{k}}$. By inserting this in the above formula, rewriting the sum as an integral and changing to a dimensionless variable we find

$$\frac{1}{N(0)V} = \int_0^{\beta_c \hbar\omega_c/2} \frac{\tanh x}{x} dx = \ln\left(\frac{2e^\gamma}{\pi}\beta_c \hbar\omega_c\right)$$

($\gamma \approx 0.577...$: the Euler constant)

Determination of T_c

Critical temperatur T_c

$$kT_c = \beta_c^{-1} \approx 1.13\hbar\omega_c e^{-1/N(0)V}$$

Determination of T_c

For small temperatures we find

$$\frac{1}{N(0)V} = \int_0^{\hbar\omega_c} \frac{d\xi}{(\xi^2 + \Delta^2)^{1/2}}$$

$$\Rightarrow \Delta = \frac{\hbar\omega_c}{\sinh(1/N(0)V)} \approx 2\hbar\omega_c e^{-1/N(0)V},$$

which shows that T_c and $\Delta(0)$ are not independent from each other

$$\frac{\Delta(0)}{kT_c} \approx \frac{2}{1.13} \approx 1.764$$

Temperature dependence of the energy gap

Rewriting again

$$\frac{1}{V} = \frac{1}{2} \sum_{\mathbf{k}} \frac{\tanh(\beta E_{\mathbf{k}}/2)}{E_{\mathbf{k}}}.$$

in an integral form and inserting $E_{\mathbf{k}}$ gives

$$\frac{1}{N(0)V} = \int_0^{\hbar\omega_c} \frac{\tanh \frac{1}{2}\beta(\xi^2 + \Delta^2)^{1/2}}{(\xi^2 + \Delta^2)^{1/2}} d\xi,$$

which can be evaluated numerically.

Temperature dependence of the energy gap

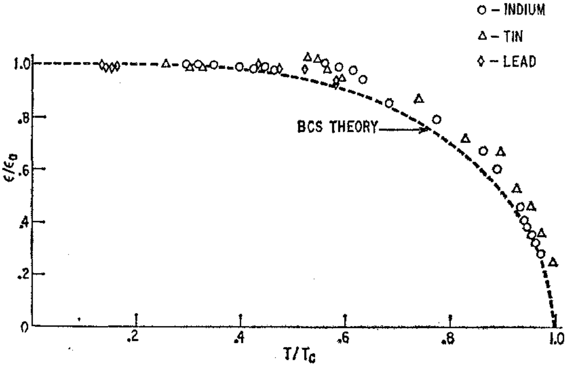


Figure: Temperature dependence of the energy gap with some experimental data (Phys. Rev. **122**, 1101 (1961))

Temperature dependence of the energy gap

Near T_c we get

Temperature dependence of Δ

$$\frac{\Delta(T)}{\Delta(0)} \approx 1.74 \left(1 - \frac{T}{T_c}\right)^{1/2}, \quad T \approx T_c,$$

which shows the typical square root dependence of the order parameter for a mean-field theory.

Thermodynamic quantities

With $\Delta(T)$ determined, we know the fermion excitation energies $E_{\mathbf{k}} = \sqrt{\xi_{\mathbf{k}}^2 + \Delta(T)^2}$. Then the quasi-particle occupation numbers will follow the Fermi-function $f_{\mathbf{k}} = (1 + e^{\beta E_{\mathbf{k}}})^{-1}$, which determine the

electronic entropy for a fermion gas

$$S_{es} = -2k \sum_{\mathbf{k}} ((1 - f_{\mathbf{k}}) \ln(1 - f_{\mathbf{k}}) + f_{\mathbf{k}} \ln f_{\mathbf{k}}).$$

Thermodynamic quantities

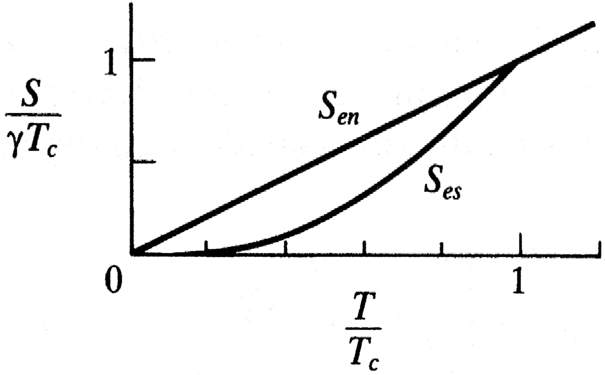


Figure: Electronic entropy in the superconducting and normal state.

Thermodynamic quantities

Given $S_{es}(T)$, we find the

specific heat

$$C_{es} = -\beta \frac{dS_{es}}{d\beta} = 2\beta k \sum_{\mathbf{k}} -\frac{\partial f_{\mathbf{k}}}{\partial E_{\mathbf{k}}} \left(E_{\mathbf{k}}^2 + \frac{1}{2}\beta \frac{d\Delta^2}{d\beta} \right)$$

In the normal state we have

$$C_{en} = \frac{2\pi^2}{3} N(0) k^2 T.$$

Thermodynamic quantities

We expect a jump in the specific heat from the superconducting to the normal state:

$$\Delta C = (C_{es} - C_{en})|_{T_c} = N(0) \left(\frac{-d\Delta^2}{dT} \right) \Big|_{T_c} \approx 9.4 N(0) k^2 T_c$$

Thermodynamic quantities

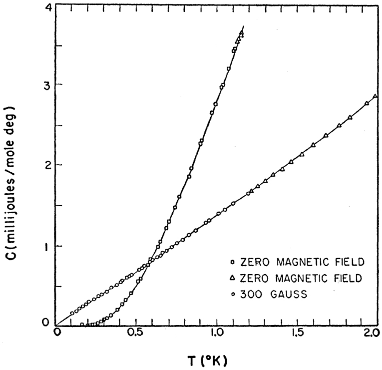


Figure: Experimental data for the specific heat in the superconducting and normal state (Phys. Rev. **114**, 676 (1959))

Type I superconductors

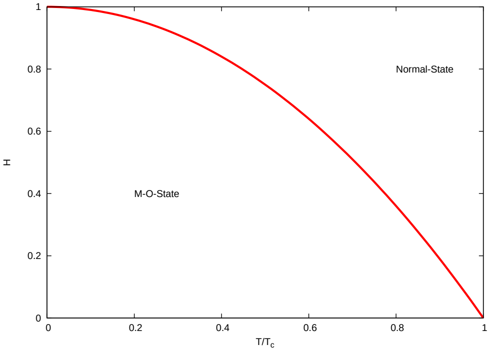
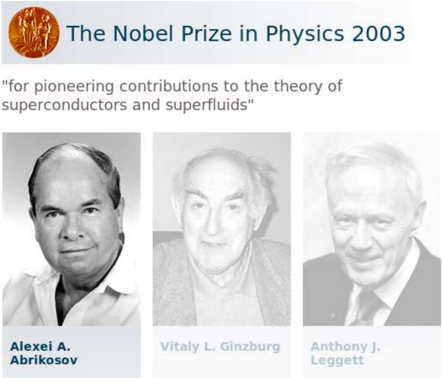


Figure: Phase diagram of a Type I superconductor

Vortex-State



Original publication: Zh. Eksperim. i Teor. Fiz. **32**, 1442 (1957) (Sov. Phys. - JETP **5**, 1174 (1957))

Type I and Type II superconductors

By applying Ginzburg-Landau theory for superconductors one finds two characteristic lengths:

1. The Landau penetration depth for external magnetic fields λ and
2. the Ginzburg-Landau coherence length ξ, which characterizes the distance over which ψ can vary without undue energy increase.

Type I and Type II superconductors

Define

Ginzburg-Landau parameter

$$\kappa \equiv \frac{\lambda}{\xi}$$

By linearizing the GL equations near T_c one can find:

- $\kappa < \frac{1}{\sqrt{2}}$: Type I superconductor
- $\kappa > \frac{1}{\sqrt{2}}$: Type II superconductor

Type II superconductors

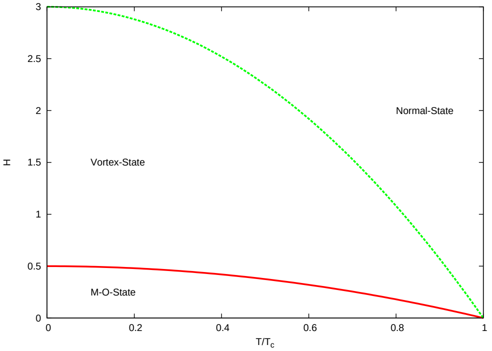


Figure: Phase diagram of a Type II superconductor

As a solution of the GL equation, one could find the following form of the orderparameter:

$$\begin{aligned}\Psi(x,y) &= \frac{1}{\mathcal{N}} \sum_{n=-\infty}^{\infty} \exp\left(\frac{\pi(ixy - y^2)}{\omega_1 \Im \omega_2} + i\pi n \right. \\ &\left. \quad + \frac{i\pi(2n+1)}{\omega_1}(x+iy) + i\pi \frac{\omega_2}{\omega_1} n(n+1) \right)\end{aligned}$$

$$\mathcal{N} = \left(\frac{\omega_1}{2\Im\omega_2} \exp\left(\pi \frac{\Im \omega_2}{\omega_1} \right) \right)^{1/4}$$

Vortex-State

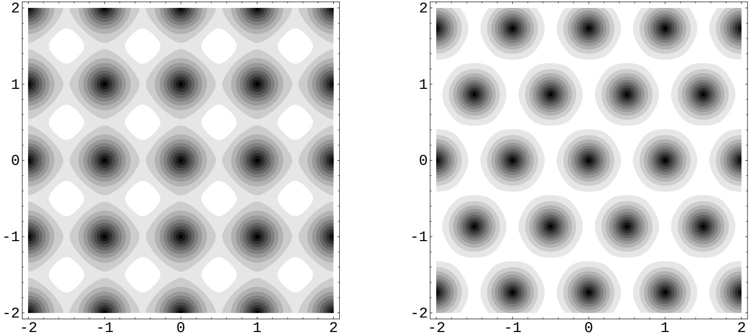


Figure: Square and triangle symmetric state of the vortex lattice in a density plot of $|\Psi|^2$.

Vortex-State

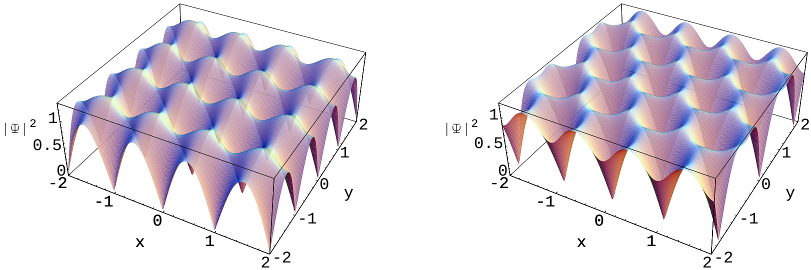


Figure: Square and triangle symmetric state of the vortex lattice in a 3D plot.

Summary

- An attractive interaction between electrons will result in forming bound Cooper pairs.
- The model-hamiltonian can be diagonalized using a Bogoliubov-Valatin-Transformation.
- The order parameter in a superconductor is the energy-gap Δ.
- BCS-Theory gives a prediction of the critical temperature T_c and the energy gap $\Delta(T)$.
- Vortices will be observed in Type II superconductors.

The END

Thank you for your attention!

Are there any questions?