III. The Bose Gas with Hard Cores in the Mean Field Limit

Hamiltonian

$$H = \sum_i \left[\frac{\vec{p}_i^2}{2m} + V(\vec{r}_i) \right] + U_0 \sum_{i<j} \delta(\vec{r}_i - \vec{r}_j)$$

Hartree Approximation

$$\Psi(\vec{r}_1, \ldots, \vec{r}_N) = \prod_{i=1}^{N} \phi(\vec{r}_i)$$

$$\delta \mathcal{F}[\phi] \stackrel{!}{=} 0$$

$$\mathcal{F}[\phi] = \langle \Psi | H | \Psi \rangle - \mu N \langle \phi | \phi \rangle$$

Hartree-Fock Approximation

$$\Psi = \frac{N_1! \cdots N_k!}{N!} \sum_{[\pi] \in S_n} \prod_{i_1=1}^{N_1} \phi_{\varepsilon_1}(\vec{r}_{\pi^{-1}(i_1)}) \cdots \prod_{i_k=N_1+\cdots+N_{k-1}+1}^{N_1+\cdots+N_k} \phi_{\varepsilon_k}(\vec{r}_{\pi^{-1}(i_k)})$$

Calculation of $\mathcal{F}$ and its Variation

$$\mathcal{F}[\phi] = N \int \mathrm{d}\vec{r}\, \phi^*(\vec{r}) \left(\frac{\vec{p}^2}{2m} + V(\vec{r}) + \frac{1}{2} U_0 (N-1) |\phi(\vec{r})|^2 - \mu \right) \phi(\vec{r})$$

$$\delta\mathcal{F}[\phi] = N \int \mathrm{d}\vec{r} \left(\delta(\phi^*) \frac{\vec{p}^2}{2m} \phi(\vec{r}) + \cdots - \mu \phi^*(\vec{r}) \delta(\phi) \right)$$

Case

$$\delta\phi = 0$$

$$\frac{\vec{p}^2}{2m}\phi(\vec{r}) + V(\vec{r})\phi(\vec{r}) + U_0(N-1)|\phi(\vec{r})|^2\phi(\vec{r}) - \mu\phi(\vec{r}) = 0$$

Wavefunction of the Condensed State

$$\psi = \sqrt{N}\phi \qquad N - 1 \approx N$$

Gross-Pitaevski Equation

$$-\frac{\hbar^2}{2m}\triangle\psi + V\psi + U_0|\psi|^2\psi = \mu\psi$$

Thomas-Fermi Approximation

$$n(\vec{r})U_0 + V(\vec{r}) = \mu$$

Properties of the Harmonic Trap

Boundary of the Cloud

$$V(\vec{r}) = \mu$$

$$R_i = \sqrt{2\mu/m\omega_i^2}$$

Chemical Potential and Number of Particles

$$N = \int_{V(\vec{r})\leq\mu} \mathrm{d}\vec{r}\, n(\vec{r}) = \frac{8\pi}{15}\left(\frac{2\mu}{m\overline{\omega}^2}\right)^{\frac{3}{2}}\frac{\mu}{U_0}$$

$$\mu \propto N^{\frac{2}{5}}$$

$$\frac{E}{N} = \frac{5}{7}\mu$$