II. The Non-Interacting Bose Gas

Thermodynamic Potential

$$\Omega_\nu = -k_B T \log \sum_{n_\nu=0}^{\infty} e^{n_\nu(\mu-\varepsilon_\nu)/k_B T}$$

Convergence

$$\mu < 0$$

Bose Distribution Function

$$n_\nu = -\frac{\partial \Omega_\nu}{\partial \mu} = \frac{1}{e^{\frac{\varepsilon_\nu - \mu}{k_B T}} - 1}$$

Chemical Potential

$$N = \sum_\nu (e^{(\varepsilon_\nu-\mu)/k_B T} - 1)^{-1}$$

Transition Temperature

$$N = N_{ex}(T_C, \mu = 0)$$

Harmonic Trap

$$V(\vec{r}) = \sum_{i=1}^{3} \frac{1}{2} m\omega_i^2 r_i^2$$

Energy Levels

$$\varepsilon_{(n_1,n_2,n_3)} = \sum_{i=1}^{3} (n_i + 1/2)\hbar\omega_i$$

$$N^{1/3} \gg 1 \quad T \gg \hbar\omega_i/k_B$$

Number of States

$$\frac{1}{\Pi_{i=1}^{3} \hbar\omega_i} \int_0^{\varepsilon} \mathrm{d}\varepsilon' \int_0^{\varepsilon-\varepsilon'} \mathrm{d}\varepsilon'' \int_0^{\varepsilon-\varepsilon'-\varepsilon''} \mathrm{d}\varepsilon''' \, 1 = \frac{1}{6} \frac{\varepsilon^3}{\Pi_{i=1}^{3} \hbar\omega_i}$$

Density of States

$$(2 \prod_{i=1}^{3} \hbar\omega_i)^{-1} \varepsilon^2 = c\varepsilon^2$$

implying

$$N_{ex}(T_C, \mu = 0) \approx \int_0^{\infty} \mathrm{d}\varepsilon \, c\varepsilon^2 \frac{1}{e^{\varepsilon/k_B T_C} - 1} = c(k_B T_C)^3 \Gamma(3)\zeta(3)$$

Transition Temperature

$$T_C \approx \frac{\hbar}{k_B} \left(\frac{\omega_1\omega_2\omega_3 N}{\zeta(3)} \right)^{\frac{1}{3}} \approx 0.94 \frac{\hbar\overline{\omega}}{k_B} N^{1/3}$$

Number of Particles in the Condensate

$$N_0 = N - N_{ex}(T, \mu = 0) \approx N(1 - (\frac{T}{T_C})^3)$$

Specific Heat

$$T < T_C \ (\mu = 0)$$

$$E \approx \int_0^\infty \mathrm{d}\varepsilon \, c\varepsilon^2 \frac{\varepsilon}{e^{\varepsilon/k_B T} - 1} = c\Gamma(4)\zeta(4)(k_B T)^4$$

$$C_v = \left(\frac{\partial E}{\partial T}\right)_v \approx 4\frac{E}{T}$$

$$E \approx 3\frac{\zeta(4)}{\zeta(3)} N k_B \frac{T^4}{{T_C}^3}$$

$$C_v \approx 12\frac{\zeta(4)}{\zeta(3)} N k_B \left(\frac{T}{T_C}\right)^3$$

$$T > T_C \ (\mu \neq 0)$$

$$N \approx \int_0^\infty \mathrm{d}\varepsilon \, c\varepsilon^2 \frac{1}{e^{(\varepsilon-\mu)/k_B T} - 1}$$

$$E \approx \int_0^\infty \mathrm{d}\varepsilon \, c\varepsilon^3 \frac{1}{e^{(\varepsilon-\mu)/k_B T} - 1}$$

Discontinuity of the Specific Heat at $T = T_C$

$$\frac{\mathrm{d}E}{\mathrm{d}T} = (\frac{\partial E}{\partial T})_\mu + (\frac{\partial E}{\partial \mu})_T \frac{\partial \mu}{\partial T}$$

$$\mu = \partial\mu/\partial T|_{T=T_C}(T - T_C)\ (T \geq T_C)$$

$$\Delta C|_{T=T_C} = (\frac{\partial E}{\partial \mu})_T \frac{\partial \mu}{\partial T}|_{T=T_C}$$

On one hand ...

$$(\frac{\partial E}{\partial \mu})_T \stackrel{p.I.}{=} 3N$$

... while on the other ...

$$(\frac{\partial \mu}{\partial T})_N = -(\frac{\partial N}{\partial T})_\mu (\frac{\partial N}{\partial \mu})_T^{-1}$$

$$(\frac{\partial N}{\partial T})_\mu \stackrel{p.I.}{=} \int_0^\infty \mathrm{d}\varepsilon \frac{2c\varepsilon\frac{\varepsilon-\mu}{T} + c\varepsilon^2\frac{1}{T}}{e^{\frac{\varepsilon-\mu}{k_B T}} - 1} \xrightarrow{(T\to T_C,\mu\to 0)} \frac{3N}{T_C}$$

$$(\frac{\partial N}{\partial \mu})_T \stackrel{p.I.}{=} \int_0^\infty \mathrm{d}\varepsilon \frac{2c\varepsilon}{e^{\frac{\varepsilon-\mu}{k_B T}} - 1} \xrightarrow{(T\to T_C,\mu\to 0)} \int_0^\infty \mathrm{d}\varepsilon \frac{2c\varepsilon}{e^{\frac{\varepsilon}{k_B T_C}} - 1} = \frac{\zeta(2)}{\zeta(3)} \frac{N}{kT_C}$$

$$\Delta C|_{T=T_C} = -9N\frac{\zeta(3)}{\zeta(2)}k \approx -6.58Nk$$

$$\text{Density of States} \propto \varepsilon^{\alpha-1}$$

$$\Delta C \propto \frac{\zeta(\alpha)}{\zeta(\alpha-1)}$$

Homogeneous Bose Gas

$$\Delta C \longrightarrow 0 \text{ as } \alpha \searrow 2$$

$$\Delta(\frac{\partial C}{\partial T})_v \cong -3.66\frac{N}{T_C}$$