O(N) Model and 1/N Expansion

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Introduction

O(N) model: N-dim spins in lattice of arbitrary dimensions
O(N) refers to invariance of Hamiltonian under O(N) group

- Exactly solvable for $N \rightarrow \infty$

Goal: Find thermodynamic behaviour for physical N
How : by expanding in 1/N away from exact theory

- Focus on large N limit

Outline

1. The Model
2. Mathematical Background
   - Weak Coupling Expansion (brief)
   - Steepest Descent Method
3. Symmetry Breaking (leading order)
   - Effective Action
   - Dimension d = 1,2,3,4
4. 1/N Expansion (main steps)
5. Renormalization and Critical Exponents
   - Calculations at leading order
   - Results for next-to-leading order
6. Concluding Remarks.

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1. **The Model**
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The Model I

Ginzburg-Landau-Wilson Model:

$$\mathcal{H}(\phi^a, \partial_\mu \phi^a) = \overbrace{\frac{1}{2}(\partial_\mu \phi^a)^2 + \frac{1}{2}\mu_0^2 \phi^a \phi^a}^{1} + \overbrace{\frac{\lambda_0}{8N}(\phi^a \phi^a)^2}^{2} ; \text{ a} = 1,\dots,\text{N}; \ \mu = 1,\dots,\text{d}$$

- $\mathcal{H}$: essentially Landau free energy for N-dim spins
- Field-Theoretical view:
  - 1: Klein-Gordon (μ_0 bare mass)
  - 2: Interaction ($\lambda_0 > \mathbf{0}$ bare coupling constant)

The Model II

Ensemble: T $=$ const; magnetic field components $J^a =$ const.
Therefore

$$\mathcal{Z}[\beta, J^a] = \mathcal{N} \int \mathcal{D}\phi^a \exp\left(- \int \left[\mathcal{H} - J^a(x)\phi^a(x) \right] d^d x \right).$$

Introduce

$$\mathcal{W} \equiv \ln \mathcal{Z}[\beta, J^a]$$

From either $\mathcal{Z}$ or $\mathcal{W}$ all the thermodynamic quantities can be calculated:
How?

Outline

Mathematical Background

Two strategies

1. Weak Coupling Expansion
2. Steepest Descent Method

Keep in mind: we are looking for expansion in $1/N$

References

L. H. Ryder, *Quantum Field Theory*, Cambridge University Press, 2005

J. Zinn-Justin, *Path Integrals in Quantum Mechanics*, Oxford University Press, 2005

Weak Coupling Expansion

Achieved by Taylor expanding the interaction term in $\mathcal{Z}$:

$$\mathcal{Z}[\beta, J^a] = \exp\left[-\int d^d x V\left(\frac{\delta}{\delta J^a(x)}\right)\right]\mathcal{Z}_0[\beta, J^a]$$

where $V(\phi^a) = \frac{\lambda_0}{8N}(\phi^a\phi^a)^2$ and $\mathcal{Z}_0$ corresponds to $\lambda_0 = 0$.

This method

- provides expansion in $\frac{\lambda_0}{8N}$
- does not take into account number N of spin components

$\longrightarrow$ **leading terms in 1/N are infinite**

Steepest Descent Method I

$$\mathcal{Z}[\beta, J^a] = \mathcal{N} \int \mathcal{D}\phi^a \exp\left(-\int \left[\mathcal{H} - J^a(x)\phi^a(x)\right] d^d x\right).$$

Assumptions:

- $\phi^a \phi^a \sim N$
- $J^a J^a \sim N$

$\therefore \mathcal{H} \sim N$

and $\mathcal{H} - J^a \phi^a$ is a functional of the general form $\mathcal{A}/\kappa$
where $\kappa \equiv 1/N$

$\therefore \mathcal{Z}[\kappa] = \int \mathcal{D}\phi^a \exp\left[-\int d^d x \mathcal{A}(\phi^a(x))/\kappa\right]$

Steepest Descent Method II

For simplicity $\mathcal{A} \equiv A(x)$ where $x \in \mathbb{R}$ and $\mathcal{Z} \to \mathcal{I}$

$$\mathcal{I}(\kappa) = \int_a^b dx \exp\left(-A(x)/\kappa\right);$$

require $A(x)$ real analytic function in $[a, b]$

Approximate $\mathcal{I}(\kappa)$ for $\kappa \to 0^+$
$\therefore$ look for absolute minimum of A: $A(x_c)$ with $x_c \in (a, b)$.
For vanishingly small κ

$$\mathcal{I}(\kappa) \approx \mathcal{I}_\varepsilon(\kappa) = \exp\left(-A(x_c)/\kappa\right) \int_{x_c-\varepsilon}^{x_c+\varepsilon} dx \exp(-A''(x_c)x^2/\kappa)$$

where ε sufficiently small.
Note: Actual range of integration $\sim 1/\sqrt{\kappa}$

Steepest Descent Method III

Change of variables $x \longmapsto y \equiv (x - x_c)/\kappa$

$$A/\kappa = A(x_c)/\kappa + \frac{1}{2}A''(x_c)y^2 + \frac{1}{6}\sqrt{\kappa}A'''(x_c)y^3 + \frac{1}{24}\kappa A^{(4)}(x_c)y^4 + O(\kappa^{3/2})$$

At leading order in κ: $\mathcal{I} = \mathcal{I}_\varepsilon$.
Let range of integration $\rightarrow \infty$

$$\mathcal{I}(\kappa) \approx \sqrt{2\pi\kappa/A''(x_c)}\,\exp\left(-A(x_c)/\kappa\right)$$

Include all orders by expanding the exponential that multiplies the gaussian measure:

$$\mathcal{I}(\kappa) = \sqrt{2\pi\kappa/A''(x_c)}\,\exp\left(-A(x_c)/\kappa\right)\mathcal{J}(\kappa)$$

where

$$\mathcal{J}(\kappa) = 1 + \sum_{n=1}^{\infty} J_n\kappa^n; \qquad J_n = \text{gaussian averages.}$$

This is exactly the 1/N expansion we were looking for!!!

Outline

Symmetry Breaking

Aim: Establish whether spontaneous symmetry breaking occurs at leading order in 1/N for 1,2,3,4 space(lattice) dimensions

$\therefore$ look for $\langle\phi^a\rangle_{J^a}$ in the thermodynamic limit, as $J^a \to 0$.

For conventional purposes rename:

$\langle\phi^a\rangle_{J^a} \equiv \phi^a$
and
$\langle\phi^a\rangle_{J^a\to 0} \equiv \langle\phi^a\rangle$

References
L.H. Ryder, *Quantum Field Theory*, Cambridge University Press, 2005
S.Coleman, R.Jackiw and H.D. Politzer, Phys. Rev. **D10**, 2491-2499 (1974)
L.F. Abbott, J.S. Kang and H. J. Schnitzer, Phys. Rev. **D13**, 2212-2226 (1976)

The Effective Action I

Define effective action Γ (Legendre transformation of $\mathcal{W}$):

$$\Gamma[\phi^a] = \mathcal{W}[J^a] - \int d^d x \phi^a(x) J^a(x)$$

$$\therefore \qquad \left.\frac{\delta \Gamma}{\delta \phi^a(x)}\right|_{\langle \phi^a \rangle} = 0$$

$\rightarrow$ Equilibrium in vanishing external field corresponds to extremum of Γ

Demand uniform distribution of spins at equilibrium (i.e. $\phi^a(x) = const$)

Define effective potential $V(\phi^a)$:

$$V(\phi^a) \equiv -\frac{1}{\Omega}\Gamma(\phi^a)$$

where Ω is the volume of space (lattice)

The Effective Action II

Still have to check whether extremum is maximum or minimum!
Calculate the Hessian of V at $\langle \phi^a \rangle$:

$$\frac{\partial^2 V}{\partial \phi^a \partial \phi^b} = -\Gamma^{(2)}_{ab}(\vec{p}_i = 0) = \left(G^{(2)}_{ab}(\vec{p}_i = 0) \right)^{-1} \equiv M^2_{ab}$$

where M^2_{ab} is the squared mass tensor.

$$\begin{aligned} \therefore\ M^2_{ab} &\geqslant 0 \text{ : equilibrium} \\ &< 0 \text{ : no equilibrium} \longrightarrow \text{tachyons} \end{aligned}$$

Note: $M^2_{ab} \longleftrightarrow (\chi_{ab})^{-1}$; χ_{ab} = susceptibility

Effective Potential at leading order I

Introduce auxiliary field χ then at leading order

$$V(\phi^a, \chi) = -\frac{N}{2\lambda_0}\chi^2 + \frac{1}{2}\chi\phi^2 + \frac{N\mu_0^2}{\lambda_0}\chi + \frac{1}{2}N\int \frac{d^d k}{(2\pi)^d} \ln(k^2 + \chi)$$

The extrema of V are now determined by the following two equations:

$$\frac{\partial V}{\partial \chi} = 0 \text{ and } \frac{\partial V}{\partial \phi^a} = 0$$

Effective Potential at leading order II

$$\frac{\partial V}{\partial \chi} = 0 \rightarrow$$

$$\phi^2 = \frac{2N}{\lambda_0}\chi - \frac{2N\mu_0^2}{\lambda_0} - N\int \frac{d^d k}{(2\pi)^2}\frac{1}{k^2+\chi}$$

yielding

$$\frac{dV}{d\phi^2} = \frac{\partial V}{\partial \phi^2} + \frac{\partial V}{\partial \chi}\frac{\partial \chi}{\partial \phi^2} = \frac{1}{2}\chi$$

$$\frac{\partial V}{\partial \phi^a} = 0 \rightarrow$$

$$\frac{dV}{d\phi^2}\phi^a = 0$$

$\therefore$ **symmetry breaking** if $\exists\ \phi^2 > 0$ with $\chi = 0$.

Dimension d=1

Theory non divergent
$\mu^2 = \mu_0^2$ and $\lambda = \lambda_0$

$$\phi^2 = \frac{2N}{\lambda}\chi - \frac{2N\mu^2}{\lambda} - \frac{N}{2\sqrt{\chi}}$$

Recall: look for positive ϕ^2 at $\chi = 0$. Here impossible!

$\therefore$ minimum at $\phi^2 = 0$

No symmetry breaking for d = 1!

Dimension d=2

Need for renormalization:

$$\frac{\mu^2}{\lambda} \equiv \frac{\mu_0^2}{\lambda_0} + \frac{1}{2}\int \frac{d^2k}{(2\pi)^2}\frac{1}{k^2+M^2} \text{ and } \lambda \equiv \lambda_0$$

where M^2 is a regularization parameter with dimensions of mass squared.

We derive

$$\phi^2 = \frac{2N}{\lambda}\chi - \frac{2N\mu^2}{\lambda} + \frac{N}{4\pi}\ln(\chi/M^2)$$

for any positive value of M^2 no $\phi^2 > 0$ for $\chi = 0$.

No symmetry breaking for d = 2!

Dimension d=3

Need for renormalization:

$$\frac{\mu^2}{\lambda} \equiv \frac{\mu_0^2}{\lambda_0} + \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\frac{1}{k^2} \text{ and } \lambda \equiv \lambda_0$$

yielding

$$\phi^2 = \frac{2N}{\lambda}\chi - \frac{2N\mu^2}{\lambda} + \frac{N}{4\pi}\sqrt{\chi};$$

demand $\lambda > 0$ (because $\lambda_0 > 0$).

Two cases:

1. $\mu^2 \geqslant 0$: minimum at $\phi^2 = 0$
2. $\mu^2 < 0$: positive ϕ^2 for $\chi = 0$ at

$$\phi^2 = -\frac{2N\mu^2}{\lambda}$$

To the right of the minimum V monotonically increasing and convex.
To the left of the minimum V complex: unphysical.

Symmetry breaking exists for d = 3!

Dimension d=4 I

Requires a more careful analysis.
Coleman et al. (1974) predict symmetry breaking, but analysis of correlation functions reveals appearence of tachyons!
Dilemma resolved by Abbott et al. (1976).

Required renormalization:

$$\frac{\mu^2}{\lambda} \equiv \frac{\mu_0^2}{\lambda_0} + \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2},$$

$$\frac{1}{\lambda} = \frac{1}{\lambda_0} + \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2(k^2+M^2)}$$

$$\therefore \qquad \chi = \mu^2 + \frac{\lambda}{2}\left(\frac{\phi^2}{N}\right) + \frac{\lambda}{32\pi^2}\chi \ln(\chi/M^2)$$

Dimension d=4 II

Renormalization invariant quantities **identified:**

- μ^2/λ
- $\chi_0 \equiv M^2 \exp(32\pi^2/\lambda)$.

Define ρ through: $\chi(\phi^2) = \rho(\phi^2)\chi_0$

The equation $\dfrac{\partial V}{\partial \chi} = 0$ takes the form

$$\rho \ln \rho = -\frac{32\pi^2}{\chi_0}\left(\frac{\mu^2}{\lambda}\right) - \frac{16\pi^2}{\chi_0}\left(\frac{\phi^2}{N}\right)$$

Dimension d=4 III

Existence of branch point! ϕ_b:

$$\frac{\phi_b^2}{N} = \frac{\chi_0\, e^{-1}}{16\pi^2} - 2\left(\frac{\mu^2}{\lambda}\right)$$

such that

- $\forall\ \phi^2 > \phi_b^2$: $\qquad ImV(\phi^2) \neq 0$
- $\forall\ 0 < \phi^2 < \phi_b^2$: V double valued function of ϕ^2

Further analysis shows that for any value of $\frac{\mu^2}{\lambda}$ the **symmetric branch is below the asymmetric one**.

No symmetry breaking for d = 4!

Outline

1/N expansion I

Goal: calculate critical exponents in three dimensions

Rewrite our Hamiltonian as:

$$\mathcal{H}(\phi^a, \partial_\mu \phi^a) = \frac{1}{2}(\partial_\mu \phi^a)^2 + \frac{1}{2} t_0 \phi^a \phi^a + \frac{\lambda_0}{8N}(\phi^a \phi^a)^2; \text{ a} = 1,\ldots,\text{N}; \; \mu = 1,\ldots,\text{d}$$

with

$$\mu_0^2 \longrightarrow t_0$$

Reference

I.D. Lawrie and D.J. Lee, Phys. Rev. **B64**, 184505:1-11 (2001)

1/N expansion II

Rewrite partition function using a Hubbard-Stratanovich transformation:

$$\mathcal{Z}[J^a] = \mathcal{N} \int \mathcal{D}\Psi \exp\left[-NH_{eff} + \frac{1}{2} \int d^3r \int d^3r' \, J^a(\mathbf{r})\Delta(\mathbf{r},\mathbf{r'};\Psi)J^a(\mathbf{r'}) \right]$$

where

$$H_{eff}[\Psi] = \int d^3r \frac{1}{\lambda_0}\Psi^2(\mathbf{r}) - \frac{1}{2} Tr \ln \Delta(\mathbf{r},\mathbf{r'};\Psi)$$

and Δ (propagator of ϕ field) defined by

$$[-\nabla^2 + t_0 + i\Psi(\mathbf{r})]\Delta(\mathbf{r},\mathbf{r'};\Psi) = \delta(\mathbf{r}-\mathbf{r'})$$

1/N expansion III

Introduce inverse susceptibility $\tilde{t}_0 \equiv -\Gamma^{(2)}(0) = \left(G^{(2)}(0)\right)^{-1}$ and expand about $\langle \Psi(\mathbf{r}) \rangle$.

Apply steepest descent method:
change variables $\Psi \to \psi$:

$$\Psi(\mathbf{r}) = -i\left(\tilde{t}_0 - t_0 - \frac{1}{N}\delta\right) + N^{-1/2}\psi(\mathbf{r})$$

and require $\langle \psi(\mathbf{r}) \rangle = 0$

1/N expansion IV

After enough hard work... (use Feynman diagrams and see report)
From $\langle\psi(\mathbf{r})\rangle = 0 \rightarrow$ **constraint equation**:

$$\begin{aligned} t_0 &= \tilde{t}_0 - \frac{\lambda_0}{2}\int \frac{d^3k}{(2\pi)^3}\Delta(\mathbf{k}) \\ &+ N^{-1}\left[\frac{\lambda_0}{8}A(\tilde{t}_0,\lambda_0) - \lambda_0\Sigma(\mathbf{0};\tilde{t}_0,\lambda_0)D(0)^{-1}\right] \\ &+ O(N^{-2}) \end{aligned}$$

where $\Sigma(\mathbf{p};\tilde{t}_0,\lambda_0) = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\Delta(\mathbf{k}+\mathbf{p})D(\mathbf{k})$
and $A(\tilde{t}_0,\lambda_0) = \int \frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}\Delta(\mathbf{k})^2\Delta(\mathbf{k'})D(\mathbf{k}+\mathbf{k'})$

1/N expansion V

...and **correlation functions**

$$\Gamma^{(2)}(\mathbf{p}) = p^2 + \tilde{t}_0 + N^{-1}[\Sigma(\mathbf{p}; \tilde{t}_0, \lambda_0) - \Sigma(\mathbf{0}; \tilde{t}_0, \lambda_0)]$$

and

$$\Gamma^{(4)}(\mathbf{0}) = N^{-1} D(\mathbf{0}) + O(N^{-2})$$

Outline

Renormalisation & critical exponents at leading order I

At leading order **constraint equation** yields

$$t_0 - t_{0_c} = \tilde{t}_0 + 2a\lambda_0 \tilde{t}_0^{1/2}$$

where

$$t_{0_c} = -\frac{\lambda_0}{2}\int \frac{d^3k}{(2\pi)^3}\frac{1}{k^2} \text{ and } a \equiv 1/(16\pi).$$

Apply the most general renormalization scheme:

$$\begin{aligned} \lambda_0 &= mZ_\lambda(\lambda)\lambda \\ t_0 - t_{0_c} &= m^2 Z_t(\lambda) t \\ \tilde{t}_0 &= m^2 \tilde{t} \end{aligned}$$

Reference

I.D. Lawrie and D.J. Lee, Phys. Rev. **B64**, 184505:1-11 (2001)

Renormalisation & critical exponents at leading order II

Apply normalization conditions

$$\begin{aligned} \Gamma^{(2)}(p^2 = 0, t = 1) &= m^2 \\ \lim_{N\to\infty} N\Gamma^{(4)}(\mathbf{p}_i = 0, t = 1) &= m\lambda \end{aligned}$$

which **define** the renormalization functions Z_λ and Z_t:

$$\begin{aligned} Z_\lambda(\lambda) &= (1 - a\lambda)^{-1} = \frac{z + a}{z} \\ Z_t(\lambda) &= \frac{1 + a\lambda}{1 - a\lambda} = \frac{z + 2a}{z} \end{aligned}$$

where $z \equiv \lambda^{-1} - a$

Renormalization & critical exponents at leading order III

Consider

$$(t_0 - t_{0_c})(\lambda_0, \tilde{t}_0) = m^2 \left(1 + \frac{2a}{z}\right) t(z, \tilde{t})$$

$\therefore$ **renormalization-group equation**

$$\left[z\frac{\partial}{\partial z} - 2\tilde{t}\frac{\partial}{\partial \tilde{t}} + 2 - \frac{2a}{z+2a}\right] t(z, \tilde{t}) = 0$$

By solving this equation we obtain the **scaling relations**: **How?**

Renormalization & critical exponents at leading order IV

Method of Characteristics consider following diff. eq.

$$a(x,y)u_x + b(x,y)u_y = c(x,y)$$

define: $F(x,y,z) = u(x,y) - z = 0$
$\therefore$ **characteristic equations:**

$$\frac{dx}{dl} = a(x,y); \quad \frac{dy}{dl} = b(x,y); \quad \frac{dz}{dl} = c(x,y)$$

plug back in our functions and obtain:

$$z(l) = zl; \quad \tilde{t}(l) = \tilde{t}l^{-2}; \quad t(z(l),\tilde{t}(l)) = l\left(\frac{zl+2a}{z+2a}\right)$$

defining $\tilde{t}(l) = 1$, critical point corresponds to $l \to 0$

$$\therefore \qquad \tilde{t}(l) \sim \tilde{t}l^{-2-\eta} = \tilde{t}l^{-2} \qquad \text{and } t(l) \sim tl^{1/\nu} = tl$$

Results for critical exponents

Our calculated values (leading order):

$$\eta = 0; \qquad \nu = 1; \gamma = (2-\eta)\nu = 2$$

Results at next-to-leading order:

$\eta = 8(3\pi^2 N)^{-1} + O(N^{-2})$

$\nu = 1 - 32(3\pi^2 N)^{-1} + O(N^{-2})$

$\gamma = 2(1 - 12(\pi^2 N)^{-1}) + O(N^{-2})$

Outline

Concluding Remarks

What have we learnt?

Methods

- to identify symmetry breaking
- to develop $1/N$ expansion
- to calculate critical exponents using RG