Path Integrals I

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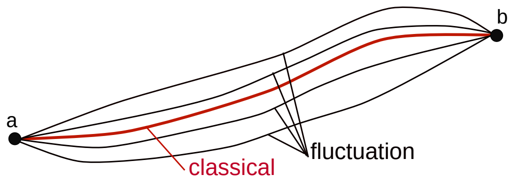
The operator formalism of quantum mechanics is not always the most transparent way of understanding quantum phenomena. Path integrals are a much more demonstrative way and, unlike the differential character of the Schrödinger equation, constitute an all-time approach in calculating quantum mechanical amplitudes to go from a to b.

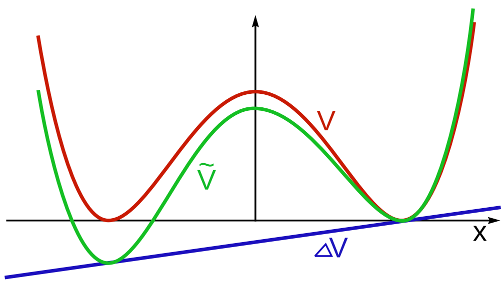
We begin by deriving the path integral for a single point particle, starting with the time displacement operator $\hat{U}(t_b, t_a)$. Using the fundamental composition law of this operator, time is sliced into infinitesimal pieces, where we can further calculate the time displacement amplitude $(x_b t_b | x_a t_a) \equiv \langle x_b | \hat{U}(t_b, t_a) | x_a \rangle$. Taking the continuum limit, we obtain the path integral.
Based on the original form, we go on to path integrals in quantum statistics and therefore introduce the important concept of imaginary time. This enables us to express the quantum statistical partition function with path integrals.

The semiclassical approach is now applied to the path integral, which leads to a general formula, dividing classical action and fluctuation terms.
As a first application we regard the quantum harmonic oscillator and exactly calculate its lowest order fluctuation term.



In tunneling problems, there is no classical solution for $E_{kin} < V_{max}$ and the concept of the semiclassical approach for path integrals therefore is not directly applicable here. To solve this issue, we introduce imaginary time for single particle situations which makes the potential V flip to $-V$ and allows us to find a classical path.

Having introduced this general methods for the use of path integrals, we then turn to an important application:



The double well potential V constitutes a non-trivial but solvable problem that has some real-world applications, for example the inversion doubling of the ammonia molecule. In addition, this calculations enable us to derive the lifetime of unstable states.
Beginning with the approximation of two independent harmonic oscillators, we calculate the energy shift of the ground states which is caused by a non-vanishing tunneling probability through the barrier in the middle. We use the semiclassical approach for the path integral of the system and calculate the contribution of the classical path, as well as the fluctuations.

To extend our results to the decay of states, we only have to make some small changes, i.e. we add a small linear term $\triangle V$ to the double well potential to receive $\tilde{V}$. Receiving an imaginary energy for the lowest state now leads to a finite lifetime.