Mean field III
Hubbard-Stratonovich transformation and saddlepoint approximation

David Moser

The application of the mean field approximation within the formalism of coherent state path integrals is introduced. The advantage of this approach compared to the operator formalism mean field approximation is a more intuitive insight into the meaning of the approximation and the physics involved. Also it offers ways to go beyond the mean field approximation by estimating the stability of the mean field.
In the first part, the general physical and mathematical concepts involved are shown.
The meaning of mean value in statistical mechanics and the implementation of the mean field approximation in operator formalism are reconciliated. Basic features of coherent states and the corresponding path integrals are discussed. The partition function is formulated as a path integral, which is the starting point for the mean field approximation.
A general interaction Hamiltonian is considered and inserted into the partition function. Now the Hubbard-Stratonovich transformation is applied, which corresponds to choosing a mean field. This transformation decouples the interaction part of the Hamiltonian by substituting a gaussian integral for the quartic terms. This yields a new path integral over an auxiliary field which is real or complex. The auxiliary field is the order parameter of the problem and closely linked to the mean field.
Now the fermions can be integrated out leaving only a path integral over the auxiliary field. There the actual mean field approximation takes place when carrying out a saddlepoint approximation. The saddlepoint equation corresponds to the self-consistency equation of the operator mean field formalism. In the end, one can estimate the stability of the mean field by analyzing quadratic terms in the exponent, the gaussian fluctuations of the mean field.
In the second part the framework acquired before is applied to RPA and Superconductivity.
RPA will be revisited with an emphasis on the implementation of the mean field approximation and the physical results being charge screening and plasma oscillations.
Superconductivity is described in greater detail. First, the attractive interaction is justified and the corresponding Hamiltonian derived. Then, the order parameter with its physical properties are introduced and the mean field approximation is performed. Using the resulting partition function the temperature dependency of the order parameter is derived.