WKB-Theory - The Quasiclassical Approximation

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The quasiclassical approximation to the static solution of the one-dimensional Schrödinger equation will be presented. It is also known as WKB theory, in reference to Wentzel, Kramers, and Brillouin, who popularized it. To obtain the approximation, the ansatz $\Psi = A \cdot \exp\left(\frac{i}{\hbar} S\right)$ is made, and then S is expanded as a series in powers of $\hbar$. With the intuitive idea in mind, that $\hbar \ll 1$ for the case of classical physics, all terms $O(\hbar^2)$ are then neglected.

We see that the quasiclassical approximation breaks down at points, where the energy E of the solution equals the potential V (classical turning points), and therefore an alternative approximation has to be made locally: By assuming V to be locally linear in the turning points of the problem, the solution can be expressed in terms of Airy functions. In an area where both the quasiclassical solution, as well as the linear approximation of the potential are applicable, the two found solutions can be matched. By this comparison of the exact solution of the linearized problem and the quasiclassical one, a phase shift for the WKB solution in the classically allowed region appears. Combining turning points will then lead us to quantum wells and tunneling. In quantum wells, the matching of the two solutions propagating from each of the turning points, yields a quantization condition, from which the energy levels of the quantum well may be calculated. For the tunneling problem, new boundary conditions have to be imposed. By comparing incoming and outgoing waves of the solution, the quasiclassical coefficients of transmission and reflection can established. Unfortunately the reflection coefficient turns out to be 1 for all tunnels. For quasiclassical treatment of inhomogeneous equations finally, the Green's function of the Schrödinger equation will be approximated.

The WKB method will then be illustrated using some examples: The calculation of the energy levels of the radial coulomb equation, and the discussion of the quasiclassical scattering, in particular scattering of protons on hydrogen atoms. For the coulomb potential, if the solutions are assumed to have an angular momentum $l \neq 0$ however, the quasiclassical approximation is not valid near $r = 0$. Fortunately, this problem can easily be solved by a variable transformation. In terms of the new variables the coulomb problem may indeed be solved quasiclassically. Rephrasing the new quantization condition in terms of the old variables, yields the known quantization condition with $l(l+1)$ replaced by $(l+\frac{1}{2})^2$. This is known as the Langer correction of the angular momentum. The coulomb problem for solutions with $l = 0$ on the other hand, has no linear turning point in $r = 0$, and therefore the phase shift has to be obtained by comparison with the analytic solution. The resulting quasiclassical energy levels agree with the one's obtained by exact calculation. For quasiclassical scattering, we derive the formal expression the differential scattering cross section by using the WKB ansatz and considering the equation of continuity. In the radial part of the solution, again a turning point phase shift has to be taken in account. These results, can then be compared with the exact results.