Optimization & Physics
Exercise 7

Maximum Flow methods can be applied to study random field spin models. Our goal is to determine the ground states of a random field Ising model (RFIM) by the maximum flow method.
The idea is to construct an equivalent network, such that the maximum flow through the network corresponds to the ground state energy of the RFIM.

Problem 7.1 Maximum Flow: Ford-Fulkerson and Edmonds-Karp algorithm

After having expressed the transformation of the random field Ising model onto a network in the previous exercise, the present exercise adheres to the implementation of the Ford-Fulkerson algorithm for the random field Ising model.
The RFIM is described by the Hamilonian:

$$\mathcal{H} = -\sum_{\langle ij \rangle} J S_i S_j - \sum_i B_i S_i, \tag{1}$$

where $J > 0$ denotes the magnitude of the ferromagnetic interaction and the spins $S_i \in \{+1, -1\}$, $i = 1, ..., N$ are arranged on a two dimensional cubic lattice. A local random field B_i acts on each spin and is drawn by a bimodal distribution

$$p_{\pm} = 0.5\delta(B - \Delta) + 0.5\delta(B + \Delta), \tag{2}$$

where the parameter Δ is a measure for the strength of the disorder.
The idea of the Ford-Fulkerson algorithm is to keep searching for a path, along which the flow can be increased. If we cannot push additional flow, described by the set $\{f_{ij}\}$, from the source to the sink any longer, the maximum flow is found. Use the concept of the residual network with residual capacities $\{r_{ij}\}$ to determine the maximum flow of the network. A detailed description of the Ford-Fulkerson algorithm is given in the lecture notes.
As it has been mentioned in the lecture, the Ford-Fulkerson algorithm may unnecessarily get stuck in long loops. To avoid this unfavorable behavior, you are required to implement the Edmonds-Karp algorithm, which is an extension of the Ford-Fulkerson algorithm. The basic idea is to choose a path, which has the minimal number of edges between the source and the sink. Using a breadth-first-search automatically gives you a shortest path, you just need to make sure that the chosen path has available capacity, i.e. the minimal capacity of the residual network is nonzero.
After having found the maximum flow, you need to translate it to a minimal cut, in order to get the ground-state. To do so, you start at the source and continue along edges for which $f_{ij} < c_{ij}$ to stay on the source side of the minimal cut. The visited vertices belong to S, the other vertices to $\bar{S}$ and have the opposite spin direction.