Optimization & Physics
Exercise 6

Maximum Flow methods can be applied to study random field spin models. Our goal is to determine the ground states of a random field Ising model (RFIM) by the maximum flow method in the next two exercises.
The idea is to construct an equivalent network, such that the maximum flow through the network corresponds to the ground state energy of the RFIM. As a first step you are required to implement the mapping of a diluted random bond and random field model onto a network and as a second step you will have to implement a maximum flow algorithm, which delivers the ground states of the RFIM.

Problem 6.1 Maximum Flow: Mapping onto a network

The RFIM is described by the Hamilonian:

$$\mathcal{H} = -\sum_{\langle ij\rangle} J S_i S_j - \sum_i B_i S_i, \tag{1}$$

where $J > 0$ denotes the magnitude of the ferromagnetic interaction and the spins $S_i \in \{+1, -1\}$, $i = 1, ..., N$ are arranged on a two dimensional cubic lattice. A local random field B_i acts on each spin and is drawn by a bimodal distribution

$$p_{\pm} = 0.5\delta(B - \Delta) + 0.5\delta(B + \Delta), \tag{2}$$

where the parameter Δ is a measure for the strength of the disorder.
The RFIM is a special case of the diluted random field model given by the Hamiltonian:

$$\mathcal{H} = -\sum_{i<j} J_{ij}\epsilon_i\epsilon_j S_i S_j - \sum_i B_i \epsilon_i S_i, \tag{3}$$

where $\epsilon_i \in \{0, 1\}$ determines, whether or not a site is occupied by a spin and the bonds $J_{ij} > 0$ can be of variable strengths.
For the transformation onto a network, we consider a network $N = (G, c, s, t)$, where $G = (V, E)$ is a directed graph which has N+2 vertices and c are the capacities of the edges $(i, j) \in E$, with $c_{ij} = 0$, if $(i, j) \notin E$. The vertices $s = 0, t = N + 1 \in V$ are the source and the sink of the network. We divide the network in two parts $(S, \bar{S})$, such that the source and the sink are in different parts. The capacity $C(S, \bar{S})$ of the cut is given by:

$$C(S, \bar{S}) = \sum_{i \in S, j \in \bar{S}} c_{ij}, \tag{4}$$

which is minimal, if the flow going through the network is maximal. The cut can be represented by a vector $\mathbf{X} = (x_0, ..., x_{n+1})$ with $x_i = 1$ if $i \in S$ and $x_i = 0$ if $i \in \bar{S}$. A spin configuration can now be mapped onto a cut by defining $x_i = 0.5(S_i + 1)$. By comparing the Hamiltonian of a diluted random field model with the expression for the capacity or by seeking advice in the lecture notes, you can find out how the capacities have to be expressed in terms of the bonds, the dilution coefficients and the local random fields. Your task is to provide a code for the transformation, which for a given realization $\{J_{ij}\}$, $\{\epsilon_i\}$ and $\{B_i\}$ specifies the capacities c_{ij} as well for the function, which returns the capacitiy $C(S, \bar{S})$ of a given cut.